Relationship-preserving Riemann investigation

Wizard Joe's Millennium Rainbow · Paul Cooper's proposed construction
Mathematical review prepared 14 October 2026. Status: rigorous reductions and obstruction results; RH remains unresolved.

This manuscript develops the construction in the supplied brief. Its governing instruction is to preserve the relationships before reducing the description. The correlation, reflected zero partner, prime powers, real-place term, endpoints, complex coefficients, and S13 provenance remain explicit throughout. It proves the support-preserving parametrization and the exact compact-test positivity equivalence. It also proves why several attractive positivity shortcuts fail. It supplies an independently recomputed numerical probe and a separate interval certificate for the complete three-Gaussian span, with an unconditional tail bound. Neither finite result is promoted to unrestricted RH.

1. Statement and conventions

For $\operatorname{Re} s > 1$ put

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} = \prod_{p} (1 - p^{-s})^{-1}.$$

The series and Euler product converge absolutely there. Meromorphic continuation has its only pole at $s = 1$, simple with residue one. Define

$$\Lambda_\zeta(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s), \qquad \xi(s) = \tfrac{1}{2}s(s-1)\Lambda_\zeta(s), \qquad \Xi(t) = \xi(\tfrac{1}{2} + it).$$

The first is meromorphic; ξ and Ξ are entire. Their symmetries are

$$\xi(s) = \xi(1-s), \qquad \xi(\bar{s}) = \overline{\xi(s)}, \qquad \Xi(-t) = \Xi(t).$$

For real t, $\Xi(t)$ is real. Bombieri's supplied paper uses $\xi(t)$ for what we denote $\Xi(t)$; the variable and convention must not be silently interchanged. The nontrivial zeros form a multiset $\mathcal{Z}$ in $0 < \operatorname{Re}\rho < 1$, with their analytic multiplicities. They are the zeros of ξ. The trivial zeta zeros $-2, -4, \ldots$ are excluded from $\mathcal{Z}$ and are accounted for by the gamma factor in the explicit formula. RH says $\operatorname{Re}\rho = 1/2$ for every $\rho \in \mathcal{Z}$.

The functional equation and conjugation give the same-multiplicity reflected partner

$$\rho^* = 1 - \bar{\rho}.$$

It coincides with ρ exactly when $\operatorname{Re}\rho = 1/2$.

All multiplicative Mellin transforms below use the measure dx/x:

$$\hat{f}(s) = \int_0^\infty f(x) x^s \, \frac{dx}{x}.$$

The correlation uses the different, deliberate measure dy:

$$f_{a,b}(x) = \int_0^\infty a(xy)\overline{b(y)}\,dy, \qquad f_g = f_{g,g}.$$

Unless explicitly declared otherwise, the exact test space is $\mathcal{D} = C_c^\infty(0,\infty;\mathbb{C})$. Write

$$\mathcal{D}_0 = \{g \in \mathcal{D} : G(0) = G(1) = 0\}.$$

The Gaussian family used later belongs to a different, rapidly decaying class. It is not compactly supported.

2. Correlation, reflected partner, and convergence

Theorem 1 (Mellin factorization). If $a, b \in \mathcal{D}$ have transforms A, B, then $f_{a,b} \in \mathcal{D}$ and

$$\hat{f}_{a,b}(s) = A(s)\overline{B(1-\bar{s})} \qquad (s \in \mathbb{C}).$$

Proof. Suppose the supports lie in positive compact intervals $[a_0, a_1]$ and $[b_0, b_1]$. The correlation vanishes outside $[a_0/b_1, a_1/b_0]$. Differentiation under the integral gives smoothness; each derivative is integrated over the same compact y interval. On the support of the integrand, x, y, xy are bounded and bounded away from zero. For each fixed s, the double integral is absolutely integrable, so Fubini and $u = xy$ give

$$\begin{aligned}
\hat{f}_{a,b}(s) &= \int_0^\infty \overline{b(y)} \left[\int_0^\infty a(xy)x^s \,\frac{dx}{x}\right] dy \\
&= \int_0^\infty \overline{b(y)}y^{-s}\,dy \int_0^\infty a(u)u^s\,\frac{du}{u} \\
&= A(s)\overline{B(1-\bar{s})}.
\end{aligned}$$

Compact support also permits differentiation in s to all orders, so all the transforms are entire. This proves the identity without any zero-location assumption. □

For one test, define the complete Weil form

$$Q(g) = \sum_{\rho \in \mathcal{Z}} G(\rho)\overline{G(\rho^*)}.$$

To justify its convergence, set $\phi(u) = e^{u/2}g(e^u)$. Then

$$G(s) = \int_{\mathbb{R}} \phi(u)e^{(s-1/2)u}\,du.$$

Repeated integration by parts gives $G(\beta + i\gamma) = O_m((1+|\gamma|)^{-m})$, uniformly for $0 \le \beta \le 1$, for

every positive integer m. The classical unconditional zero count is $N(T) = O(T\log(T+2))$. Consequently the defining sum for Q is absolutely convergent. Multiplicity is retained and no summation order matters for these smooth tests. For Bombieri's more general piecewise-smooth class, use his symmetric-height convention $\lim_{T\to\infty}\sum_{|\operatorname{Im}\rho|<T}$.

Reindexing by $\rho \mapsto \rho^*$ gives $\overline{Q(g)} = Q(g)$. More generally,

$$K_{ij} = \sum_{\rho} G_j(\rho)\overline{G_i(\rho^*)}, \qquad K_{ji} = \overline{K_{ij}}, \qquad Q\left(\sum_j c_j g_j\right) = c^* K c.$$

This proves Hermitian symmetry, not positivity. A finite zero subset yields a Hermitian truncated matrix only when it is closed under the reflected-partner involution, including multiplicities.

Why the reflected partner is indispensable. For an off-line pair, write $G(\rho) = a$ and $G(\rho^*) = b$. Their exact contribution is $a\bar{b} + b\bar{a} = 2\operatorname{Re}(a\bar{b})$. Replacing it by $|a|^2 + |b|^2$ changes the form by $|a-b|^2$. With $a = 1, b = -1$, the two quantities are -2 and $+2$. This is an exact algebraic counterexample to the replacement. It is not an actual off-line zeta zero or an RH counterexample.

3. Complete arithmetic formula

Write $F = \hat{f}$, let γ_E be Euler's constant, and let $\Lambda(n) = \log p$ when $n = p^k$ with p prime and $k \geq 1$, and zero otherwise. In the conventions above, Bombieri's explicit formula is

$$\begin{aligned}\sum_{\rho} F(\rho) = F(0) + F(1) - \sum_{n=2}^{\infty}\Lambda(n)\left(f(n) + \frac{1}{n}f(1/n)\right) \\ - (\log(4\pi) + \gamma_E)f(1) - \int_1^{\infty}\frac{f(x) + x^{-1}f(1/x) - 2x^{-1}f(1)}{x - x^{-1}}\,dx.\end{aligned}$$

For $f = f_g$, its left side is exactly $Q(g)$ by Theorem 1. All signs, the factor $1/n$, the subtraction $2f(1)/x$, and the constant $\log(4\pi) + \gamma_E$ are essential. The endpoint term is

$$F(0) + F(1) = G(0)\overline{G(1)} + G(1)\overline{G(0)} = 2\operatorname{Re}(G(0)\overline{G(1)}).$$

It vanishes for $g \in \mathcal{D}_0$. It is not $|G(0)|^2 + |G(1)|^2$.

Origin of the terms. On a vertical line $\operatorname{Re} s = c > 1$, the absolutely convergent logarithmic derivative is $-\zeta'(s)/\zeta(s) = \sum_n \Lambda(n)n^{-s}$. Apply Mellin inversion on that line and on its reflected line $1-c$, then shift the contour. The residues at the nontrivial zeros have their analytic multiplicities, the pole at one and its completed reflected endpoint supply $F(1)$ and $F(0)$, and the two Dirichlet-series integrals give $f(n)$ and $f(1/n)/n$. The logarithmic derivative

$$\frac{\xi'(s)}{\xi(s)} = \frac{1}{s} + \frac{1}{s-1} - \frac{1}{2}\log\pi + \frac{1}{2}\frac{\Gamma'(s/2)}{\Gamma(s/2)} + \frac{\zeta'(s)}{\zeta(s)}$$

accounts for every remaining contribution. Evaluating its gamma term, for example with the integral

representation of the digamma function and keeping its two cancelling terms together at zero, gives precisely the displayed constant and integral. The trivial zeros are absorbed into this gamma term; adding a second trivial-zero sum would double count them. Smooth compact support gives rapid decay on vertical lines, so the horizontal contour terms tend to zero along zero-avoiding heights. Bombieri's stated class includes these tests: they and their derivatives are smooth and vanish near both endpoints. Thus the theorem applies directly, without an extension by numerical analogy.

The real-place calculation, including the constant. Put $C(u) = e^{u/2} f(e^u)$ and $\mathcal{F}(t) = F(1/2 + it) = \int C(u) e^{itu} du$. Fourier inversion gives $(2\pi)^{-1} \int \mathcal{F}(t) dt = C(0) = f(1)$. The gamma contribution on the central line is

$$J = \frac{1}{2\pi} \int_{\mathbb{R}} \mathcal{F}(t) \operatorname{Re} \psi_\Gamma(1/4 + it/2)\, dt,$$

where $\psi_\Gamma = \Gamma'/\Gamma$. The elementary digamma representation gives

$$\operatorname{Re} \psi_\Gamma(1/4 + it/2) = -\gamma_E + \int_0^\infty \frac{e^{-v} - e^{-v/4} \cos(tv/2)}{1 - e^{-v}}\, dv.$$

With $v = 2u$ and Fourier inversion of the cosine factor, this becomes

$$J = -\gamma_E C(0) + \int_0^\infty \frac{2e^{-2u} C(0) - e^{-u/2}(C(u) + C(-u))}{1 - e^{-2u}}\, du.$$

Here the numerator terms are kept together at zero; near zero their difference is controlled using the second Fourier moment, and at infinity exponential decay suffices. Since $\mathcal{F}$ is Schwartz, these bounds justify Fubini and the limiting operations. Finally

$$2 \int_0^\infty \frac{e^{-2u} - e^{-u}}{1 - e^{-2u}}\, du = -2 \int_0^\infty \frac{e^{-u}}{1 + e^{-u}}\, du = -2 \log 2.$$

Combining J with the $-\log \pi\, C(0)$ conductor term therefore gives

$$-(\log(4\pi) + \gamma_E) C(0) - \int_0^\infty \frac{e^{-u/2}(C(u) + C(-u)) - 2e^{-u} C(0)}{1 - e^{-2u}}\, du,$$

which, after $x = e^u$, is exactly the real-place expression above. This calculation fixes the constant and sign without numerical calibration.

The integral is an ordinary improper integral with a removable endpoint, not a Cauchy principal value. Its numerator is zero at $x = 1$. Differentiating numerator and denominator there gives $f(1)$ and 2, respectively, hence the endpoint value is $f(1)/2$. At infinity, once $f(x)$ and $f(1/x)$ vanish, its tail is exactly $-2f(1)/(x^2 - 1)$, which is integrable. If $R > 1$ lies beyond the correlation support and its reciprocal support, then

$$\int_R^\infty \frac{-2f(1)}{x^2-1}\,dx = f(1)\log\frac{R-1}{R+1}.$$

This often-forgotten real-place tail remains even when the test is compactly supported.

For stable numerical calculation set $x = e^u$ and

$$C_g(u) = e^{u/2} f_g(e^u) = \int_{\mathbb{R}} \phi(v+u)\overline{\phi(v)}\,dv.$$

Then $C_g(-u) = \overline{C_g(u)}$ and the integral becomes

$$\int_0^\infty \frac{2e^{-u/2}\operatorname{Re} C_g(u) - 2e^{-u}C_g(0)}{1-e^{-2u}}\,du.$$

At $u = 0$ use the exact limit $C_g(0)/2$. Evaluate the numerator as

$$2e^{-u/2}(\operatorname{Re} C_g(u) - C_g(0)) + 2C_g(0)e^{-u}\operatorname{expm1}(u/2)$$

and the denominator as $-\operatorname{expm1}(-2u)$ to reduce cancellation. Numerical stabilization does not supply an interval error bound by itself.

4. Exact support-preserving parametrization

Let $D = x\,d/dx$ and $L = D(D+1) = x^2 d^2/dx^2 + 2x\,d/dx$.

Theorem 2 (exact test-space correspondence). The map $L : \mathcal{D} \to \mathcal{D}_0$ is a bijection. For every compact interval $[a,b] \subset (0,\infty)$ it is also a bijection between the corresponding subspaces with support contained in $[a,b]$. Its inverse is

$$h(x) = \int_0^x \frac{g(t)}{t}\,dt - \frac{1}{x}\int_0^x g(t)\,dt.$$

Proof. Differentiation preserves support. Integration by parts, with both boundary terms zero because h vanishes near zero and infinity, gives

$$\widehat{Dh}(s) = -sH(s), \qquad \widehat{(D+1)}h(s) = (1-s)H(s).$$

Therefore $G(s) = s(s-1)H(s)$, and in particular $G(0) = G(1) = 0$.

Conversely set $A(x) = \int_0^x g(t)dt/t$ and $B(x) = \int_0^x g(t)dt$. These are smooth on $(0,\infty)$. The proposed $h = A - B/x$ obeys

$$h'(x) = \frac{B(x)}{x^2}, \qquad h''(x) = \frac{g(x)}{x^2} - \frac{2B(x)}{x^3}, \qquad x^2h'' + 2xh' = g.$$

If $x < a$ then $A = B = 0$. If $x > b$, the two moment conditions give $A = G(0) = 0$ and $B = G(1) = 0$. Hence h vanishes off $[a, b]$. It is smooth across the endpoints: the integral formula is smooth on all of $(0, \infty)$, and it agrees there with zero on the exterior open sets. Equivalently, the original smooth compactly supported g and all its derivatives vanish to every order at each support boundary, as do the resulting derivatives of h. No boundary delta or derivative jump is introduced.

Finally, $Lh = 0$ implies $(x^2 h')' = 0$, so $h = c_0 + c_1/x$ on the connected positive half-line. Compact support forces $c_0 = c_1 = 0$. This proves uniqueness and the exact support-interval correspondence. Support can fill interior gaps of the original support; the statement is interval containment, not equality of every disconnected support component. □

5. The restricted compact-test criterion, proved

Theorem 3. The following statements are equivalent:

1. RH holds.
2. $Q(g) \geq 0$ for every $g \in \mathcal{D}$.
3. $Q(g) \geq 0$ for every $g \in \mathcal{D}_0$.
4. $Q(Lh) \geq 0$ for every $h \in \mathcal{D}$.

This is the applicable Weil criterion in the exact test space of the construction. Bombieri states the related moment-restricted criterion for his broader class. The argument below proves the compact smooth restriction, rather than assuming that a smaller class automatically suffices.

Proof. RH gives $\rho^* = \rho$; the absolutely convergent sum is then a sum of $|G(\rho)|^2$. Thus 1 implies 2, which implies 3. Theorem 2 makes 3 and 4 equivalent. It remains to prove 4 implies 1.

Write $\lambda_\rho = \rho - 1/2$ and $p(s) = s(s-1)$. Define a distribution on compact smooth tests χ on $\mathbb{R}$ by

$$T(\chi) = \sum_\rho p(\rho)^2 \int_{\mathbb{R}} \chi(u) e^{\lambda_\rho u}\, du.$$

This distribution is well defined. On each fixed compact support, repeated integration by parts makes the integral decay faster than any chosen inverse power of $|\operatorname{Im}\rho|$, uniformly in the bounded real part. The factor $p(\rho)^2$ grows at most polynomially of degree four, and $N(T) = O(T\log(T+2))$. A sufficiently high fixed derivative seminorm therefore bounds the sum on that support.

For $h \in \mathcal{D}$, set $\psi(u) = e^{u/2} h(e^u)$, which ranges over all $C_c^\infty(\mathbb{R})$. If $C_\psi(u) = \int \psi(v+u)\overline{\psi(v)}dv$, Theorem 1 gives

$$Q(Lh) = T(C_\psi).$$

The square on $p(\rho)$ is deliberate: $\overline{p(\rho^*)} = p(1-\rho) = p(\rho)$. Replacing it by $|p(\rho)|^2$ away from the line would again discard the reflected relationship. Assumption 4 says that $T(C_\psi) \geq 0$ for every compact smooth ψ.

Choose a real, even, smooth compactly supported function η and set

$$S(x) = T(C_\eta(\cdot - x)).$$

This is a smooth positive-definite function. Indeed, for arbitrary finite coefficients c_j and translations x_j, apply the assumed positivity to $\psi(u) = \sum_j c_j \eta(u - x_j)$. Expanding its full correlation, including all mixed terms, gives

$$\sum_{j,k} c_j \overline{c_k} S(x_j - x_k) \geq 0.$$

The two-by-two case first gives the Hermitian relation $S(-x) = \overline{S(x)}$, and then its principal minors give $S(0) \geq 0$ and $|S(x)| \leq S(0)$ for every real x. This elementary boundedness step requires neither RH nor a preselected self-adjoint spectrum.

Put $E(z) = \int \eta(u) e^{zu} du$. Since η is real and even, its correlation transform at z is $E(z)^2$. Consequently

$$S(x) = \sum_\rho a_\rho e^{\lambda_\rho x}, \qquad a_\rho = p(\rho)^2 E(\lambda_\rho)^2.$$

The coefficients decay faster than any fixed inverse power of the ordinate after allowing for the polynomial factor; the sum and all its derivatives converge uniformly on compact x intervals. For $\operatorname{Re} z > 1/2$, absolute convergence permits integration term by term:

$$\int_0^\infty e^{-zx} S(x)\, dx = \sum_\rho \frac{a_\rho}{z - \lambda_\rho}.$$

The left side is holomorphic throughout $\operatorname{Re} z > 0$, because S is bounded. The right side is meromorphic there: zeros of the nonzero entire function ξ are locally finite, and the partial-fraction series converges locally uniformly away from their translated pole locations, by the rapid coefficient decay and zero-count bound.

Suppose a zero ρ_0 has $\operatorname{Re} \rho_0 > 1/2$. Select a nonnegative real even bump with integral one, scaled to sufficiently small support. Its $E(\lambda_{\rho_0})$ tends to one as the support shrinks, and is therefore nonzero. Also $p(\rho_0) \neq 0$. The right side then has a genuine pole at $z = \lambda_{\rho_0}$, with nonzero residue equal to the multiplicity of ρ_0 times $p(\rho_0)^2 E(\lambda_{\rho_0})^2$. Distinct zeros cannot cancel this residue because they have distinct pole locations. The two analytic expressions agree on $\operatorname{Re} z > 1/2$ and therefore throughout the connected right half-plane with those poles removed. A nonzero pole contradicts holomorphy of the bounded function's Laplace transform. Thus no zero lies to the right of the line. Reflection excludes zeros to the left. This proves RH. □

This theorem proves an equivalence. It does **not** establish its positivity hypothesis. In particular the boundedness of S was derived from the very positivity that remains to be proved, so it cannot be fed back as an unconditional new bound.

6. Positivity attempts and exact obstruction results

6.1 The differential parametrization is not a positive factorization

In logarithmic coordinates, L becomes $\partial_u^2 - 1/4$:

$$e^{u/2}(Lh)(e^u) = (\partial_u^2 - 1/4)\psi(u).$$

On the critical line this has the real Fourier multiplier $-(t^2 + 1/4)$. That observation explains the convenient moment factor but says nothing about off-line zero pairs. The actual spectral weight after applying L is $p(\rho)^2$, whose use in Theorem 3 preserved all off-line possibilities. Declaring $Q(Lh) = \|Th\|^2$ without independently constructing such a T would assert the desired positivity rather than prove it.

6.2 Individual prime contributions have both signs, even on moment-zero tests

Fix a prime power $n \geq 2$ and put $u_0 = \log n$. Take a nonzero real bump η supported in an interval shorter than $u_0/3$, and define $\psi_\pm(u) = \eta(u) \pm \eta(u - u_0)$. Let $h_\pm(e^u) = e^{-u/2}\psi_\pm(u)$ and $g_\pm = Lh_\pm$. Both $g_\pm$ belong to $\mathcal{D}_0$ by Theorem 2. Put $\theta = (\partial_u^2 - 1/4)\eta$, which is nonzero and has the same support bound. The two disjoint pieces give exactly

$$C_{g_\pm}(u_0) = \pm\|\theta\|_{L^2}^2.$$

All other potential terms in this particular correlation vanish by support separation. Therefore the single prime-power contribution

$$2\Lambda(n)n^{-1/2}\,\mathrm{Re}\,C_{g_\pm}(\log n)$$

is strictly positive for $+$ and strictly negative for $-$. Thus positivity cannot follow by assigning a fixed nonnegative sign to each prime-power term. The complete form also has other prime powers and the real-place contribution, which this argument does not discard or evaluate. It refutes a termwise-sign proof attempt, not RH.

6.3 Positive diagonal entries do not control mixed terms

The exact Hermitian matrix

$$K = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$$

has positive diagonal but eigenvalues 3 and -1. For $c = (1, -1)^T$, $c^*Kc = -2$, whereas the diagonal-only expression is 2. This finite example is not asserted to be a zeta Weil matrix. It disproves the general inference that retaining only positive separate contributions establishes the coupled inequality.

6.4 Finite positive evidence needs an infinite-class argument

A computed positive matrix establishes at most positivity on its specified finite span, and only with a bound for arithmetic tails and numerical error. Positivity on a sequence of finite spans can imply positivity on a full test space if their union is dense in a topology in which Q is continuous and each complete form is proved nonnegative. No such exhaustion is established by the three-Gaussian probe. A positive observed eigenvalue does not supply uniform control of omitted directions.

The current remaining implication is explicit: prove the displayed **complete arithmetic** expression is nonnegative for every compact smooth g satisfying both moments, or produce an admissible g with a rigorously negative complete upper bound. No unconditional factorization, positive-operator domain construction, applicable number-field index theorem, or genuine negative certificate has been established here.

7. S13 preserved, with its exact provenance

The existing repository's S13 README labels its formula an **explicit reference realization added to that package**. It says the formulas from the user's reported new manuscript were not supplied or recovered. This manuscript preserves that statement. It does not rename the reference realization as a recovered original proof.

Its local payload and reference encoder are

$$p = (q, b, v, \alpha), \quad q \in S^3,\ b \in \mathbb{R}^5,\ v \in \mathbb{R}^3,\ \alpha \geq 0,$$

$$r = \sqrt{2 + \|b\|^2 + \|v\|^2 + \alpha^2}, \qquad \mathcal{E}(p) = \frac{(q, b, v, \alpha, 1)}{r} = (Q, L, V, A, t) \in S^{13}.$$

Its exact image is $t > 0$, $|Q|^2 = t^2$, $A \geq 0$ on the unit sphere, and its inverse is

$$\mathcal{D}(Q, L, V, A, t) = (Q/t, L/t, V/t, A/t).$$

The image has physical dimension twelve, with boundary at $\alpha = 0$; thirteen is the ambient sphere dimension. The quaternion sign, the actual endpoint spin seed, local chart pair, and ordered history are retained. The seed and history are external record fields, not secretly stored in fourteen coordinates. No global spin-lift choice is made.

Theorem 4 (S13 transfer preserves the question). Let a specified readout $\mathcal{G}$ assign an admissible Riemann test to a complete linked record. Then

$$\widetilde{Q}(z, \text{retained record}) = Q(\mathcal{G}(\mathcal{D}(z), \text{retained record}))$$

has exactly the same value and sign as the original form on the decoded record. An exact reversible change of representation neither proves nor destroys positivity. The claim is about an explicitly supplied $\mathcal{G}$; the S13 physical encoder alone does not define a map to all admissible Mellin test functions.

Proof. Substitute $\mathcal{D}(\mathcal{E}(p)) = p$. Every argument, coefficient and relation passed to Q is unchanged. If a separately declared coefficient convention stores a finite vector c by positive scaling $d = tc$, the quadratic identity is $c^*Kc = t^{-2}d^*Kd$. Since $t > 0$, the sign is preserved and every mixed term survives. □

For the matrix in Section 6.3, choose an illustrative coefficient encoding with $q = 1$, $\alpha = 0$, $v = 0$, and $b = (1, -1, 0, 0, 0)$. Then $r = 2$, $t = 1/2$, and the stored two coefficients are $(1/2, -1/2)$. Their quadratic value is $-1/2$; decoding multiplies by $t^{-2} = 4$ to recover -2. Unit spherical norm has not turned an indefinite form positive. This is a labelled coefficient demonstration, not an identification of physical stretch coordinates with zeta data.

Likewise, bounded spherical norm does not give a uniform decoded-amplitude bound. The existing reference's records $q = 1$, $b = 0$, $\alpha = 1$, $v = \pm Me_1$ have $\|\mathcal{E}(p)\| = 1$ but $t = (3 + M^2)^{-1/2} \to 0$. Their equal-weight decoded covariance has trace M^2. The inverse is regular at finite zero stretch or zero source; it is not uniformly conditioned as $t \to 0$. This is a concrete obstruction to an amplitude-control shortcut, not a Navier–Stokes blow-up result and not an RH counterexample.

8. Gaussian reproduction and analytic tail control

For $\sigma > 0$ and real t_j put

$$h_j(x) = x^{-1/2} \exp\left(-\frac{(\log x)^2}{2\sigma^2}\right) e^{-it_j \log x}, \qquad g_j = Lh_j.$$

Completing the square in the absolutely convergent Gaussian integral gives

$$H_j(s) = \sqrt{2\pi}\sigma \exp\left(\frac{\sigma^2}{2}(s - 1/2 - it_j)^2\right), \qquad G_j(s) = s(s-1)H_j(s).$$

The Gaussian and every logarithmic derivative decay faster than any fixed power of x at both zero and infinity after the $x^{-1/2}$ factor is included. All integrations by parts have vanishing boundary terms. The correlation has the same type of decay. Thus Bombieri's class hypotheses hold, and the zero sums converge absolutely by Gaussian decay in the ordinate. These are valid rapidly decaying tests, while Theorem 2's compact-support statement remains a distinct theorem.

The mixed correlations have a closed form. Set

$$a = \frac{1}{4\sigma^2}, \quad b_{ij} = \frac{t_i + t_j}{2}, \quad r_{ij}(u) = -2au - ib_{ij},$$

$$C^h_{ij}(u) = \sqrt{\pi}\sigma \exp\left(-au^2 - \frac{\sigma^2(t_j - t_i)^2}{4} - ib_{ij}u\right).$$

Applying $(\partial_u^2 - 1/4)$ to both factors of the correlation, with justified integration by parts, gives

$$C^g_{ij}(u) = \left[r_{ij}^4 - (12a + 1/2)r_{ij}^2 + 12a^2 + a + 1/16\right] C^h_{ij}(u).$$

Then $f_{ij}(e^u) = e^{-u/2}C_{ij}^g(u)$. This exact formula retains the phase and all mixed terms and avoids a nested numerical quadrature.

Using $\sigma = 0.25$, $(t_1, t_2, t_3) = (0, 14, 21)$, prime powers at most 1000, the full real-place integral, zero endpoint term, and sixty decimal digits in mpmath 1.3.0, the independent script `reproduce_probe.py` computed the eigenvalues

$$0.063225270168765803884611 3, \qquad 15208.419698350784231966 8, \qquad 139331.289053499164540704.$$

These reproduce all three supplied targets to the shown precision. The maximum matrix-entry difference from the first twenty positive numerical zeros and their conjugates was $4.282231057124 \times 10^{-55}$ in this implementation. The supplied earlier residual 2.04×10^{-56} is therefore not literally reproduced. It is of comparable working-precision scale, with no eigenvalue or normalization disagreement detected. Neither residual is a rigorous error radius. mpmath's numerical zeros and quadrature have not been treated as a completeness or interval certificate.

Theorem 5 (explicit arithmetic tail bound for this family). Fix i, j and define

$$A_{ij} = \sqrt{\pi}\sigma e^{-\sigma^2(t_j - t_i)^2/4},$$

$$B_{ij}(u) = (2au + |b_{ij}|)^4 + (12a + 1/2)(2au + |b_{ij}|)^2 + 12a^2 + a + 1/16.$$

For an integer cutoff $N \geq 2$, let $L = \log N$. If $2aL^2 + L/2 \geq 5$, then the omitted prime-power sum has the unconditional bound

$$|P_{ij} - P_{ij}^{\leq N}| \leq 2A_{ij} \int_L^\infty uB_{ij}(u)e^{-au^2+u/2}\, du.$$

Proof. The triangle inequality gives $|C_{ij}^g(u)| \leq A_{ij}B_{ij}(u)e^{-au^2}$ for $u \geq 0$. Also $\Lambda(n) \leq \log n$, and the two reciprocal contributions each have magnitude $n^{-1/2}|C_{ij}^g(\log n)|$. Bound the prime-power sum by the sum over all integers. Each monomial of $uB_{ij}(u)$ has nonnegative coefficient and degree $k \leq 5$. The logarithmic derivative of $u^k e^{-au^2-u/2}$ is $k/u - 2au - 1/2$, nonpositive for $u \geq L$ by the stated condition. The resulting envelope is decreasing in n, so its sum over integers $n > N$ is at most its integral over $x \geq N$. Substitution $x = e^u$ proves the bound. □

The bound reduces to Gaussian tail moments after completing the square $-au^2 + u/2 = -a(u - 1/(4a))^2 + 1/(16a)$. Those moments can be evaluated using erfc and the integration-by-parts recurrence. A machine-readable certified bound additionally needs outward-rounded evaluation of these quantities; ordinary mpmath evaluation alone is labelled numerical.

For the real-place tail beyond $U > 0$, a direct unconditional bound is

$$\left|\int_U^\infty \frac{2e^{-u/2}\,\mathrm{Re}\,C_{ij}^g(u)-2e^{-u}C_{ij}^g(0)}{1-e^{-2u}}du\right| \le \frac{2}{1-e^{-2U}}\left[A_{ij}\int_U^\infty B_{ij}(u)e^{-au^2-u/2}du+|C_{ij}^g(0)|e^{-U}\right].$$

This also reduces to explicit Gaussian moments. It must be added to a rigorous finite-interval quadrature error and the prime tail when certifying the arithmetic matrix. A zero-sum comparison needs its own unconditional zero-density/counting envelope over omitted heights, not a silently assumed RH tail. The independent floating-point reproduction does not itself claim such a certificate; the separate certificate below supplies one for the complete zero form on the specified span.

8.1 An unconditional tail bound that retains off-line partners

Let $M=\max_j|t_j|$. Hasanalizade, Shen and Wong, Corollary 1.2 (2021), prove for all $t\ge e$ that

$$\left|N(t)-\frac{t}{2\pi}\log\frac{t}{2\pi e}\right| \le 0.1038\log t+0.2573\log\log t+9.3675,$$

where $N(t)$ counts every nontrivial zero with $0<\mathrm{Im}\,\rho\le t$, with multiplicity. The constants and range are taken from that corrected source, not an unverified older zero-count estimate. In particular $N(t)\le t\log t$ for $t\ge 10$: use $\pi>3$, $\log\log t\le\log t$, $0.1038+0.2573<1/2$, and $9.3675<10<5\log t$ to obtain $N(t)\le(t/6+5.5)\log t\le t\log t$.

For any possibly off-line zero $\rho=1/2+\delta+i\gamma$, $|\delta|<1/2$, the actual reflected product satisfies

$$|G_j(\rho)\overline{G_i(\rho^*)}| = 2\pi\sigma^2|\rho(\rho-1)|^2 \times\exp\left(\sigma^2\delta^2-\frac{\sigma^2}{2}[(\gamma-t_i)^2+(\gamma-t_j)^2]\right).$$

For $|\gamma|\ge M$ this is at most

$$F(|\gamma|),\qquad F(t)=2\pi\sigma^2e^{\sigma^2/4}(t^2+1)^2e^{-\sigma^2(t-M)^2}.$$

The inequalities are $|\rho(\rho-1)|^2\le(\gamma^2+1)^2$ and $|\gamma-t_j|\ge|\gamma|-M$. Neither assumes RH.

Theorem 6 (complete Gaussian zero-tail enclosure). Suppose $T\ge 10$, $T>M$, and

$$\lambda=2\sigma^2(T-M)-4/T>\mu=2/(T+1).$$

Then every matrix entry omitted beyond $|\mathrm{Im}\,\rho|>T$ has magnitude at most

$$B_T=\frac{2F(T)(T+1)\log(T+1)}{1-e^{-\lambda+\mu}}.$$

Proof. For $t\ge T$, $d\log F(t)/dt=4t/(t^2+1)-2\sigma^2(t-M)\le-\lambda$. Therefore $F(T+k)\le F(T)e^{-\lambda k}$. On the unit-height bin $(T+k,T+k+1]$, the positive-ordinate zero count is at most $N(T+k+1)\le$

$(T+k+1)\log(T+k+1)$. For $v \geq T+1$, the logarithmic derivative of $v\log v$ is $1/v + 1/(v\log v) \leq 2/(T+1) = \mu$. Integrating this derivative from $T+1$ to $T+k+1$ bounds the endpoint count by $(T+1)\log(T+1)e^{\mu k}$. Sum the geometric majorant over $k \geq 0$ and double for the negative ordinates. Absolute values were taken before summation, so any cancellations and any off-line reflected pairs are safely retained in the bound. □

For a $d \times d$ Hermitian matrix the induced operator-norm tail is at most dB_T, since every entry has magnitude at most B_T. Alternatively one may add the real interval $[-B_T, B_T]$ to each entry of a matrix proved real, and certify positivity through interval leading principal determinants. The unshifted Gaussian family's exact identity $G_j(\rho^*) = \overline{G_j(\rho)}$ and reflection of the complete zero multiset make its full matrix real symmetric. This identity is a property of this explicit Gaussian transform; it does not say $\rho = \rho^*$.

8.2 Certified positivity on the complete three-Gaussian span

Theorem 7 (finite-span positive definiteness, numerically certified). For $\sigma = 1/4$ and frequencies $(0, 14, 21)$, the complete form $Q(\sum_{j=1}^{3} c_j g_j)$ is strictly positive for every nonzero $c \in \mathbb{C}^3$. This conclusion includes all zeta zeros, not only a truncated list. Its stated trust base is the analytic theorems above and the tested FLINT ball-arithmetic implementation, not a formal proof-assistant verification of that library.

Proof and computation transcript. The independent certification module first subdivides the boundary of the rational rectangle $(-1, 2) \times (-100, 100)$ into 2376 segments. For each segment it encloses the entire zeta image, along with both rational polygon endpoint values, inside a convex complex rectangle excluding zero. The actual image arc and its rational polygon chord are homotopic within that zero-free convex set. The exact rational polygon winding number, corrected for zeta's pole at one, therefore counts all zeros inside the contour with multiplicity. The count is 58.

For 29 disjoint rational intervals at positive ordinates, ball evaluations of the real function $\Xi(t)$ at the interval endpoints have opposite signs. The intermediate value theorem proves a critical-line zero in each interval. Conjugation gives 29 negative-ordinate zeros; these exhaust the all-zero count. Thus the finite region contains exactly these simple critical-line zeros, with no uncounted off-line zero or boundary zero. The library zero routine was used to propose intervals; completeness comes from the separate contour argument, and replay does not need the candidate-indexing routine.

The matrix terms for these finite zeros are evaluated on the whole certified ordinate intervals. Theorem 6 bounds every remaining matrix entry by

$$B_{100} < 1.474006435926086 \times 10^{-159}.$$

The full matrix enclosures also include an explicit outward replay margin. Interval evaluation of their leading principal determinants at 192-bit generation precision yields the following conservative decimal intervals, each checked to contain the exact dyadic certificate enclosure:

Minor	Certified containing interval
Δ_1	$[0.118667791978489122,\ 0.118667791978489124]$

Minor	Certified containing interval
Δ_2	[1357.66127938272431, 1357.66127938272433]
Δ_3	[133974898.877588557, 133974898.877588559]

All lower endpoints are positive, so Sylvester's criterion proves positive definiteness of the actual complete real symmetric matrix. It also gives positivity for arbitrary complex coefficients, because $c = a + ib$ yields $c^*Kc = a^TKa + b^TKb$. By the exact Gaussian admissibility and explicit-formula identity, the same conclusion holds for the complete arithmetic form on this span. □

The native certificate records python-flint 0.9.0 with FLINT 3.6.0. Its dyadic midpoints, radii, exact rational inputs, contour covers and zero intervals are retained in `evidence/certification-gaussian.json`; its nested finite-zero proof is also exported separately as `evidence/certification-zeros-100.json`. A separate mathematical reviewer executed the full Gaussian replay at 224 bits, rereading the actual contour and sign enclosures, and confirmed certification, fresh recomputation, and positive definiteness on the specified span. Its receipt is `independent-certificate-replay.json`. The shown decimal bounds were checked by exact rational comparisons against the stored dyadic endpoints.

This certifies the complete zero form, and the arithmetic form by its proved analytic equality. It does not certify the independently implemented prime-sum and real-place numerical quadrature as algorithms; those outputs keep their numerical-evidence labels. It does not prove positivity for tests outside this three-dimensional span, supply a density/exhaustion theorem, or prove RH. No browser result is inferred from native execution.

9. What constitutes a certificate

A valid RH refutation by zeros must prove existence, multiplicity/count completeness in its declared region, boundary exclusion, and a real-part enclosure disjoint from 1/2. Merely evaluating zeta to a small residual at an off-line input proves none of these facts. A valid refutation by this form must give an admissible exact test plus a rigorously negative upper bound for its **complete** arithmetic form, including quadrature and every omitted tail.

A finite-height verification must separately identify enough real Hardy-function sign changes and rigorously count all strip zeros up to a declared nonzero boundary height, with multiplicities. Agreement of those counts can prove finite-range RH; a list of critical-line roots alone cannot.

Python-FLINT's `arb` and `acb` support enclosure arithmetic. Inputs must enter as exact rationals or enclosing decimal balls, and radii must be retained to the output. A library's presence does not certify an unvalidated composition. Official FLINT documentation describes zeta-zero and zero-count routines; any Python binding used must be inspected and its tested version pinned. No assumption of RH may be used in a routine labelled as verifying RH. Higher working precision is a useful numerical check, not a proof of a rigorous error bound.

10. Result ledger and remaining implication

Claim	Status	Reason
Mellin cross-correlation factorization with dy	Proved	Theorem 1; compact-support Fubini and exact change of variable
Complete explicit formula convention	Established theorem applied	Bombieri pp. 8–9; all hypotheses checked here
Support-interval preserving inverse of $D(D+1)$	Proved	Theorem 2, including endpoint smoothness and uniqueness
Compact moment-zero positivity equivalent to RH	Proved equivalence	Theorem 3; elementary positive-definite smoothing and Laplace pole argument
S13 inverse and preservation of quadratic sign	Proved for declared reference	Theorem 4; exact decoder and positive scale
Three-Gaussian eigenvalue targets	Numerical evidence reproduced	Independent sixty-digit arithmetic calculation
Complete form on the three-Gaussian span	Certified positive definite	Theorem 7; all-zero contour count, interval matrix, unconditional tail, independent replay
Prime, real-place and complete zero-tail inequalities	Proved analytic bounds	Theorems 5–6 and direct denominator estimate
Termwise-prime positivity shortcut	Contradicted	Exact compact moment-zero examples of both signs
Positive diagonals imply a positive complete form	Contradicted	Exact Hermitian 2×2 counterexample
Unit S13 norm controls decoded amplitude	Contradicted	Anchor tends to zero with amplitude tending to infinity
Complete arithmetic positivity on all $\mathcal{D}_0$	Unresolved	No unconditional positivity mechanism established
Off-line actual zeta zero or negative complete certificate	Not obtained	Synthetic examples and numerical searches do not meet certificate conditions

The remaining mathematical task is the universal inequality in Theorem 3, or its rigorously certified failure for an admissible test. The proved correspondence makes that obstruction precise and preserves all the relationships needed to investigate it. It does not bypass the obstruction.

References and provenance

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- Enrico Bombieri and Jeffrey C. Lagarias, *Complements to Li's Criterion for the Riemann Hypothesis*, Journal of Number Theory 77 (1999), pp. 274–287. Author-hosted primary manuscript. Included as context for preserving precise positivity formulations, not as a substitute for Theorem 3's proof.
- Bernhard Riemann, *Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse* (1859). Clay manuscript collection and translations.
- Elchin Hasanalizade, Quanli Shen and Peng-Jie Wong, *Counting zeros of the Riemann zeta function* (2021), Corollary 1.2. Primary preprint. The paper explicitly discusses corrections to older parameter ranges; the corollary used here is unconditional for all $T \geq e$.
- FLINT, `acb_dirichlet` reference. Official documentation; official repository documentation source. The hosted reference returned HTTP 403 to this research client; its official source was available.
- Python-FLINT, version 0.9.0 online reference. Official documentation. This is a documentation observation; the independent reproduction uses mpmath 1.3.0 and makes no FLINT certification claim.
- Local preserved S13 reference: `research/linked_calculations_summary/s13/README.md` and `s13.py` in the Millennium Rainbow source checkout. The reference's own attribution and limitations are preserved in Section 7.

The attached brief is task source material. Its requested objective is not evidence that a mathematical claim has been proved. No instructions from external mathematical documents have been treated as authority to change the task or publish unsupported conclusions.